# Variational Propagation Constant Expressions for Lossy Inhomogeneous Anisotropic Waveguides

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***Abstract*—Based on reciprocal relationships for the adjoint operator, we derive a variational formulation for the propagation constant satisfying the divergence-free condition in lossy inhomogeneous anisotropic waveguides whose media tensors have all nine components. In addition, with some advantages over previous representations, two variational formulations have been derived for waveguides with the transverse part of the media tensors decoupled from the longitudinal part. However, to obtain a variational formulation for a general lossy reciprocal problem the waveguide must be bi-directional. Each of the variational expressions results in a standard generalized eigenvalue equation with the propagation constant appearing explicitly as the desired eigenvalue. The stationarity of the formulations is shown. It is also shown that for a general lossy nonreciprocal problem the variational functional exists only if the original and adjoint waveguide are mutually bi-directional.**

## I. Introduction

RECENTLY, finite element methods have been applied extensively to waveguide problems [1]. When using finite element methods, it is desirable to use a variational functional. A thorough study of the variational electromagnetic problems based on the reaction concept introduced by Rumsey [2], [3] and Harrington [4] has been presented by Chen [5]. Also, many different variational expressions for the propagation constant have been derived for each specific problem [6]–[10]. However, further investigation of variational expressions for the propagation constant is still needed. For instance, Berk's [3] and Kumagai's formulations [9], [10] for the propagation constant are restricted to lossless media. Spurious modes [11] will occur in Rumsey's [2] formulations for the propagation constant, since the divergence-free condition is not enforced in the variational expression. Davies [7] as well as Chew's formulations [8] for the propagation constant only work for media with the longitudinal part of the medium tensor decoupled from the transverse part. The Euler equation for another of Davies' formulations [6] in terms of the full magnetic field vector does not satisfy the vector magnetic field equation, as pointed out by Hoffmann [12]. Each of the above formulations has some failings. Hence, a new formulation for the propagation constant in general waveguides is needed. For generality, the new formulation must be able to be used in waveguides containing lossy inhomogeneous anisotropic media where the tensor constitutive parameters have all nine components. In addition, the divergence-free condition must be enforced in the variational formulation. In this paper, a systematic procedure for the derivation of such a formulation is presented. Additionally, two variational formulations for reciprocal waveguides with the longitudinal part of the medium tensor decoupled from the transverse part are derived. Applied in finite element methods, these formulations will have some advantages over previously published formulations in the literature since these new formulations are more general or accurate.

For a general lossless (reciprocal or nonreciprocal) problem, the variational functional always exists. For a general lossy reciprocal problem, the variational functional exists only if the waveguide is bi-directional. For a general lossy nonreciprocal problem, the variational functional exists only if the original and adjoint waveguide are mutually bi-directional. (The original and adjoint waveguide are mutually bi-directional if for each mode with the propagation constant $\gamma$ in the original waveguide there exists a mode with propagation constant $-\gamma$ for the adjoint waveguide. A waveguide is bi-directional if modes with propagation constant $\gamma$ and $-\gamma$ will always exist simultaneously for the same waveguide). An example of a lossy nonreciprocal waveguide and its corresponding variational formulation for the propagation constant has been proposed by Chen [5]. While the divergence-free condition is not enforced in Chen's formulation, the variational formulation given by our procedure will automatically satisfy the divergence-free condition. We begin with a discussion of variational formulation issues, such as the choice of inner product, variational variables (magnetic field or electric field or both electric and magnetic field; full vector, i.e., three components of the field or only the transverse component of the field), the stationarity of variational formulations for the propagation constant and the existence of the adjoint field. Next we derive the various variational expressions in the form of standard generalized eigenvalue equations where the propagation constant appears explicitly as the desired eigenvalue.

## II. Variational Formulation Issues

Consider a differential equation

$$M \cdot P = 0 \tag{1}$$

where $M$ is a linear operator for describing the underlying physical problem and $P$ is the unknown field quantity. The

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corresponding functional, $F$, can be represented as [5], [6]

$$F = \langle P^a | M \cdot P \rangle. \qquad (2)$$

In (2), $P^a$ denotes the solutions with the corresponding adjoint operator $M^a$, i.e., $P^a$ satisfies $M^a \cdot P^a = 0$. We have

$$\begin{aligned}\delta F &\equiv \langle \delta P^a | M \cdot P \rangle + \langle P^a | M \cdot \delta P \rangle \\ &= \langle \delta P^a | M \cdot P \rangle + \langle M^a \cdot P^a | \delta P \rangle.\end{aligned}$$

Hence, the first variation of the functional $F$ vanishes when $M \cdot P = 0$ and $M^a \cdot P^a = 0$. In (2), $\langle P^a | M \cdot P \rangle$ represents the inner product between $P^a$ and $M \cdot P$. In electromagnetic problems, $M$ is given by Maxwell's equations and $P$ represents the electromagnetic field. The formulation for the functional in (2) has several drawbacks.

First, direct use of the functional in (2) does not yield a desirable eigenvalue matrix equation. If we write Maxwell's equations in the form of $M \cdot P = 0$, we have for the vector magnetic field source-free wave equation

$$\nabla \times \bar{\bar{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H} - \omega^2 \bar{\bar{\mu}} \cdot \mathbf{H} = 0 \qquad (3)$$

with $\mathbf{H}$ being the magnetic field vector, $\bar{\bar{\epsilon}}$ the permittivity tensor of the medium and $\bar{\bar{\mu}}$ the permeability tensor of the medium. With the differential operator in (3), the functional

$$F = \langle \mathbf{H}^{\mathbf{a}} | \nabla \times \bar{\bar{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H} - \omega^2 \bar{\bar{\mu}}^{-1} \cdot \mathbf{H} \rangle \qquad (4)$$

will be variational about the fields that satisfy (3). Direct use of this functional in the solution of a waveguide problem gives either a matrix eigenvalue equation with frequency as the eigenvalue [13], or a quadratic eigenvalue equation with propagation constant as the eigenvalue [14]. The latter could be seen by assuming that the field has $e^{j(\omega t - \gamma z)}$ dependence, with $\gamma$ being the propagation constant. With this assumption, the functional in (4) has the form

$$\begin{aligned}F = &\langle \mathbf{H}^{\mathbf{a}} | (\nabla_t - j\gamma\hat{z}) \times \bar{\bar{\epsilon}}^{-1} \\ &\cdot (\nabla_t - j\gamma\hat{z}) \times \mathbf{H} - \omega^2 \bar{\bar{\mu}} \cdot \mathbf{H} \rangle \qquad (5)\end{aligned}$$

where $\nabla = \nabla_t - j\gamma\hat{z}$ and $\nabla_t = \hat{x}(\partial/\partial x) + \hat{y}(\partial/\partial y)$. This functional will yield a quadratic eigenvalue equation with $\gamma$ being the eigenvalue. For computational simplicity, it is desirable that the formulation should yield the standard generalized eigenvalue equation form

$$A \cdot X_\lambda - \lambda B \cdot X_\lambda = 0 \qquad (6)$$

with $\lambda = \gamma$ or $\gamma^2$ and $A$ and $B$ being either differential or matrix operators.

Second, the functional $F$ from (2) introduces another set of unknowns, $P^a$.

Third, spurious modes will occur in using the functional given in (4) [11]. This is because the vector that satisfies (3) does not automatically satisfy the divergence-free condition $\nabla \cdot \mathbf{B}$, where $\mathbf{B} = \bar{\bar{\mu}}\mathbf{H}$. In the formulations presented in this paper, the divergence-free condition will be incorporated into the variational functional to avoid the occurrence of spurious modes. Variational formulation issues will be discussed in the following:

1) *Choice of inner product—real inner product or complex inner product*: The first step to simplify the variational formulation in (2) is to find $P^a$. The choice of inner product will determine the adjoint operator $M^a$ and the corresponding adjoint solution $P^a$, and hence the variational functional $F$. The real inner product is defined as

$$\langle f | g \rangle = \int f \cdot g \, dS \qquad (7)$$

whereas the complex inner product is defined as

$$\langle f | g \rangle = \int f^* \cdot g \, dS \qquad (8)$$

with $f^*$ being the complex conjugate of $f$, and $f$ and $g$ being two arbitrary vectors. The integral is two-dimensional, since two-dimensional waveguide problems are considered. Since the operator for Maxwell's equations is hermitian in a lossless system, we would usually choose a complex inner product in the lossless case. Therefore, the operator will be self-adjoint if we choose a complex inner product. Hence, in a lossless system the adjoint solution $P^a$ will be related to $P$ by $P^a = P$. Equation (4) becomes

$$F = \langle \mathbf{H} | \nabla \times \bar{\bar{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H} - \omega^2 \bar{\bar{\mu}} \cdot \mathbf{H} \rangle. \qquad (9)$$

Many useful formulations may be deduced from (9), for example, the classical result given by Berk [3]

$$\omega^2 = \frac{\int \nabla \times \mathbf{H} \cdot \bar{\bar{\epsilon}}^{-1} \cdot \nabla \times \mathbf{H}^* \, d\Omega}{\int \mathbf{H} \cdot \bar{\bar{\mu}} \cdot \mathbf{H}^* \, d\Omega} \qquad (10)$$

where the integration can be two-dimensional or three-dimensional, depending on the specific problem. The complex inner product is also directly related to the energy concept in a lossless system.

On the other hand, a real inner product should be used in the lossy case. The reason is that phase information in the propagation constant will be lost in a complex inner product formulation. Here, the primary concern is anisotropic inhomogeneous lossy media. Therefore, a real inner product is implied if not noted otherwise.

2) *Choice of variational variables—magnetic field or electric field*: When the permeability is homogeneous, the divergence-free condition $\nabla \cdot \mathbf{B} = 0$ can be reduced to $\nabla \cdot \mathbf{H} = 0$. Hence, the normal component of magnetic field will be continuous at all points over the domain. In addition, Ampere's law in a source-free waveguide will result in the continuity of the tangential component of the magnetic field at all points over the domain. Thus, it is convenient to use the magnetic field to set up a variational functional when the permeability is homogeneous and the permittivity is inhomogeneous. On the other hand, the electric field should be used as the variational variable when the permittivity is homogeneous and the permeability is inhomogeneous. Since the former case is common, we will give the derivation of the functional in terms of magnetic field. The functional in terms of electric field can be found by duality. When

the longitudinal part of the medium tensor is coupled to the transverse part, the longitudinal component of either magnetic or electric field has to be represented by both the transverse magnetic and transverse electric field. Therefore, the variational functional must be represented in terms of both electric and magnetic field for this case.

3) *Choice of variational variables—full vector or only the transverse component*: The functional in terms of only transverse field has fewer unknowns in the resulting matrix eigenvalue equation, but results in a more complicated expression and involves additional differentiation due to the incorporation of the divergence-free condition. In this paper we will give both three-vector and two-vector variational expression forms.

4) *Variational stationarity of propagation constant* $\gamma$: A variational functional $F$, in terms of the field, does not necessarily yield a variational expression for the propagation constant $\gamma$. For example, the functional defined by
$$F = \langle A \cdot \Phi_\gamma - \gamma\Phi_\gamma | A \cdot \Phi_\gamma - \gamma\Phi_\gamma\rangle \tag{11}$$
will be stable about the true field solution of $A \cdot \Phi_\gamma - \gamma\Phi_\gamma = 0$. Since the first variation of the functional $F$ can be written as
$$\begin{aligned}\delta F =& \langle\delta(A \cdot \Phi_\gamma - \gamma\Phi_\gamma)|A \cdot \Phi_\gamma - \gamma\Phi_\gamma\rangle \\ &+ \langle(A \cdot \Phi_\gamma - \gamma\Phi_\gamma)|\delta(A \cdot \Phi_\gamma - \gamma\Phi_\gamma)\end{aligned} \tag{12}$$
we have $\delta F = 0$ when $A \cdot \Phi_\gamma - \gamma\Phi_\gamma = 0$. Since $F = 0$ when $A \cdot \Phi_\gamma - \gamma\Phi_\gamma = 0$, we have
$$\langle A \cdot \Phi_\gamma - \gamma\Phi_\gamma | A \cdot \Phi_\gamma - \gamma\Phi_\gamma\rangle = 0. \tag{13}$$
Hence,
$$\gamma^2 = \frac{\langle A \cdot \Phi_\gamma | A \cdot \Phi_\gamma\rangle}{\langle\Phi_\gamma|\Phi_\gamma\rangle}. \tag{14}$$
However, $\delta\gamma = 0$ only when $A^a \cdot \Phi_\gamma - \gamma\Phi_\gamma = 0$, i.e., $\gamma$ is variational only when $A$ is self adjoint. From the above example, the $\gamma$ expression derived from a variational formulation for the field is not necessarily variationally stable. Thus it is important to show directly the variational stability of $\gamma$.

We want to show that the following expression for $\lambda$, derived from the functional in (2) and eigenvalue equation in (6), will always be variationally stable
$$\lambda = \frac{\langle X_\lambda^a | A \cdot X_\lambda\rangle}{\langle X_\lambda^a | B \cdot X_\lambda\rangle} \tag{15}$$
where $X_\lambda$ satisfies $A \cdot X_\lambda - \lambda B \cdot X_\lambda = 0$ with $A$ and $B$ being either differential or matrix operators and $\lambda$ the eigenvalue. The functional $F$ in (2) can be written as
$$F = \langle X_\lambda^a | A \cdot X_\lambda - \lambda B \cdot X_\lambda\rangle \tag{16}$$
with
$$A^a \cdot X_\lambda^a - \lambda^a B^a \cdot X_\lambda^a = 0 \tag{17}$$
and where
$$\lambda^a = \lambda \tag{18}$$
must be satisfied. Since $F = 0$ when $A \cdot X_\lambda - \lambda B \cdot X_\lambda = 0$ is satisfied, we have the result shown in (15). To show that $F$ in (16) is variationally stable about the true field, from (16), we have
$$\begin{aligned}\delta F =& \langle \delta X_\lambda^a | A \cdot X_\lambda - \lambda B \cdot X_\lambda\rangle \\ &+ \langle A^a \cdot X_\lambda^a - \lambda B^a \cdot X_\lambda^a | \delta X_\lambda\rangle.\end{aligned} \tag{19}$$
Hence
$$\delta F = 0 \tag{20}$$
whenever (6), (17), and (18) are satisfied. Therefore, $F$ is variationally stable about the true field. To show that the expression in (15) is variationally stable, with $\lambda = (C/D)$, the following relationship is useful
$$\begin{aligned}\delta\lambda &= \delta\frac{C}{D} \\ &= \frac{C + \delta C}{D + \delta D} - \frac{C}{D} \\ &= \frac{CD + \delta CD - DC - C\delta D}{(D + \delta D)D}.\end{aligned} \tag{21}$$
Hence
$$\delta\lambda = \frac{\delta CD - C\delta D}{(D + \delta D)D}. \tag{22}$$
Keeping only first order terms, we have
$$\delta\lambda D = \delta C - \lambda\delta D. \tag{23}$$
Using (15) and (23) we have
$$\begin{aligned}\delta\lambda\langle X_\lambda^a | B \cdot X_\lambda\rangle &= \delta\langle X_\lambda^a | A \cdot X_\lambda\rangle - \lambda\delta\langle X_\lambda^a | B \cdot X_\lambda\rangle \\ &= \langle \delta X_\lambda^a | A \cdot X_\lambda - \lambda B \cdot X_\lambda\rangle \\ &\quad - \langle X_\lambda^a | A \cdot \delta X_\lambda - \lambda B \cdot \delta X_\lambda\rangle \\ &= \delta F.\end{aligned} \tag{24}$$
Since in general $\langle X_\lambda^a | B \cdot X_\lambda\rangle \neq 0$ [15], [16], we have
$$\delta\lambda = 0 \tag{25}$$
whenever (6), (17), and (18) is satisfied. Therefore, the eigenvalue (propagation constant) is variationally stable about the true field solutions.

5) *Existence of the adjoint field:* Although the functional in (16) always yields a variational expression for the eigenvalue $\lambda$, the variational expression in (16) is meaningful only when the adjoint field solution exists, as given by (17). The adjoint field equation, (17), actually is not an eigenvalue equation since it is restricted by (18), i.e., $\lambda^a = \lambda$. We can still consider (18) as an eigenvalue equation and solve for its eigenvalue. If $A$ and $B$ in (6) are not self-adjoint, in general $\lambda$ may not be a eigenvalue of (17). If $\lambda$ is not a eigenvalue of (17), the equation
$$A^a \cdot X_\lambda^a - \lambda B^a \cdot X_\lambda^a = 0 \tag{26}$$
has only a trivial solution $X_\lambda^a$. Hence, (15) and (16) will not be a variational expression in this case.

The functional in (16) works only when the eigenvalue (6) and its adjoint (17) have identical sets of eigenvalues, i.e., (18) is satisfied. This is the reason why Chen's technique [5], although intended for solving general nonself-adjoint problems, in general works best for self-adjoint operators in eigenvalue type problems, such as those for wave propagation in lossless waveguides. In lossless systems, the operator is self-adjoint using complex inner product, therefore (18) is always true. For a lossy nonreciprocal system, if the original and adjoint waveguide, where the medium tensor is transposed to that of the original waveguide, are mutually bi-directional, (18) can also be satisfied. We will give an example later in this paper. In the next section, we derive the variational expression for the propagation constant for general waveguides.

## III. Derivation of the Variational Expression for the Propagation Constant

In the discussion of two-dimensional waveguide problems, it is assumed that the current source inside the waveguide is zero and the boundary is either pec (perfect electric conductor) or pmc (perfect magnetic conductor) or at infinity. Again, the field dependence is assumed to be $e^{j(\omega t-\gamma z)}$, with $z$ the longitudinal direction. If the vector magnetic field is separated into transverse and longitudinal parts, we have

$$\mathbf{H} = \mathbf{h} + H_z\hat{z} \tag{27}$$

where $\mathbf{h} = H_x\hat{x} + H_y\hat{y}$ is the transverse component. In the same way, we have

$$\mathbf{E} = \mathbf{e} + E_z\hat{z} \tag{28}$$

The media tensor is defined in matrix form as

$$\bar{\bar{\epsilon}} = \begin{bmatrix} \bar{\bar{\epsilon}}_{tt} & \vec{\epsilon}_{tz} \\ \vec{\epsilon}_{zt} & \epsilon_{zz} \end{bmatrix} \tag{29}$$

where $\bar{\bar{\epsilon}}_{tt}$ is a transverse dyadic, $\vec{\epsilon}_{tz}$ and $\vec{\epsilon}_{zt}$ are vectors and $\epsilon_{zz}$ is a scalar. The definition for $\bar{\bar{\mu}}$ is similar. In reciprocal problems, these tensors are symmetric.

The following criteria have been adopted for the selection of the operator representing Maxwell's equations used in the variational functional:

1) the operator is represented in terms of only transverse fields,
2) the operator is self-adjoint in lossless system,
3) the operator will lead to a standard general eigenvalue equation with the propagation constant appearing explicitly as the desired eigenvalue,
4) the operator can be used in lossy, inhomogeneous, anisotropic reciprocal or nonreciprocal problems.

Let

$$\Phi_\gamma = \begin{bmatrix} \mathbf{e} \\ j\mathbf{h} \end{bmatrix}. \tag{30}$$

Maxwell's equations in operator form [15] satisfying the above criteria can be written as

$$L\Phi_\gamma - \gamma\Gamma_z\Phi_\gamma = 0 \tag{31}$$

with

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \tag{32}$$

$$L_{11} = \omega\bar{\bar{\epsilon}}_{tt} - \frac{1}{\omega}\nabla_t \times \hat{z}\frac{1}{\mu_{zz}}\hat{z}\times\nabla_t - \omega\vec{\epsilon}_{tz}\frac{1}{\epsilon_{zz}}\vec{\epsilon}_{tz} \tag{33}$$

$$L_{12} = -\nabla_t \times \frac{1}{\mu_{zz}}\hat{z}\times\vec{\mu}_{zt} - \vec{\epsilon}_{zt}\hat{z}\times\frac{1}{\epsilon_{zz}}\nabla_t \tag{34}$$

$$L_{21} = -\nabla_t \times \frac{1}{\epsilon_{zz}}\hat{z}\times\vec{\epsilon}_{zt} - \vec{\mu}_{tz}\hat{z}\times\frac{1}{\mu_{zz}}\nabla_t \tag{35}$$

$$L_{22} = \omega\bar{\bar{\mu}}_{tt} - \frac{1}{\omega}\nabla_t \times \hat{z}\frac{1}{\epsilon_{zz}}\hat{z}\times\nabla_t - \omega\vec{\mu}_{tz}\frac{1}{\mu_{zz}}\vec{\mu}_{tz} \tag{36}$$

$$\Gamma_z = \begin{bmatrix} 0 & j\hat{z}\times I \\ j\hat{z}\times I & 0 \end{bmatrix} \tag{37}$$

and $I$ being identity operator. $H_z$ and $E_z$ are related to $\mathbf{e}$ and $\mathbf{h}$ by

$$j\omega\mu_{zz}H_z = -j\omega\vec{\mu}_{zt}\cdot\mathbf{h} + \nabla_t\cdot(\hat{z}\times\mathbf{e}) \tag{38}$$

$$j\omega\epsilon_{zz}E_z = -j\omega\vec{\epsilon}_{zt}\cdot\mathbf{e} - \nabla_t\cdot(\hat{z}\times\mathbf{h}). \tag{39}$$

Premultiplying (31) with $\Gamma_z$ results in

$$A\cdot\Phi_\gamma - \gamma\Phi_\gamma = 0 \tag{40}$$

with $A = \Gamma_z \cdot L$. Note that $\Gamma_z \cdot \Gamma_z = I$. In lossless problems, (31) is self adjoint with the choice of a complex inner product. Therefore, we have

$$\gamma = \frac{\langle\Phi_\gamma|L\Phi_\gamma\rangle}{\langle\Phi_\gamma|\Gamma_z\Phi_\gamma\rangle}$$

as the variational expression for $\gamma$ in lossless problems with the choice of a complex inner product. In lossy nonreciprocal problems, we have

$$\gamma = \frac{\langle\Phi_\gamma^a|L\Phi_\gamma\rangle}{\langle\Phi_\gamma^a|\Gamma_z\Phi_\gamma\rangle}$$

as the expression for $\gamma$, which is variational only if the original waveguide and the adjoint waveguide are mutually bi-directional. In lossy reciprocal problems, we have

$$\gamma = \frac{\langle\Phi_{-\gamma}|L\Phi_\gamma\rangle}{\langle\Phi_{-\gamma}|\Gamma_z\Phi_\gamma\rangle}$$

as the expression for $\gamma$, which is variational only if the waveguide is bi-directional. The reason that $\Phi_{-\gamma}$ is the adjoint solution for lossy reciprocal problems is explained in the following paragraph.

The permittivity tensor and the permeability tensor of the media are symmetric in lossy reciprocal systems. For Maxwell's equations, as represented in (31) and (6), $A$ is self adjoint and $B^a = -B$. Here we have $A = L$, $X_\lambda = \Phi_\gamma$, $\lambda = \gamma$, and $B = \Gamma_z$. $L$ is self adjoint with the choice of a real inner product when the medium tensor is symmetric [15]. We also have $\Gamma_z^a = -\Gamma_z$ with the choice of a real inner product. The set of eigenfunctions for (6) is identical to the set of eigenfunctions for (17) with $X_\lambda^a = X_{-\lambda}$ and $\lambda^a = \lambda$. Hence, we have

$$\lambda = \frac{\langle X_{-\lambda}|A\cdot X_\lambda\rangle}{\langle X_{-\lambda}|B\cdot X_\lambda\rangle} \tag{41}$$

with the choice of a real inner product in lossy reciprocal systems.

## IV. Adjoint Solutions for Lossy Problems

In the following, the expression for the adjoint field for lossy systems will be given. The eigenmodes in the adjoint waveguide and the eigenmodes in the original waveguide satisfy the orthogonality relationship [15]

$$\langle \Gamma_z \Phi_\beta^a | \Phi_\gamma \rangle = N_{\gamma\beta}\delta_{\gamma\beta} \begin{matrix} \delta_{\gamma\beta} = 1(\gamma = \beta) \\ \delta_{\gamma\beta} = 0(\gamma \neq \beta) \end{matrix} \tag{42}$$

where $N_{\gamma\beta}$ is some constant. Assume an normalized set such that $N_{\beta\beta} = 1$. From the orthogonality relationship in (42), we have

$$|\Phi_\gamma\rangle = \sum_\beta \langle \Phi_\gamma | \Phi_\beta \rangle | \Gamma_z \Phi_\beta^a \rangle. \tag{43}$$

From (43), we have

$$|\Phi^a\rangle = \Gamma_z D^{-1} |\Phi\rangle$$

with

$$D_{ij} = \langle \Phi_i | \Phi_j \rangle \tag{44}$$

with as long as $D$ is nonsingular. In general, relationship (44) holds even for nonreciprocal (lossy, anisotropic and inhomogeneous) waveguides. However, (44) is not useful or necessary in actual numerical calculations. If the adjoint field does exist, we can represent the original field as well as the adjoint field as a linear combination of basis functions with unknown coefficients. By taking the variations with respect to the adjoint field, the resulting matrix equation will contain only the unknown coefficient of the original field. This approach is variational only if the adjoint solution exists. The existence of the adjoint solution relies on the matrix $A$ in (44) being nonsingular, which is not always known a priori.

For reciprocal problems, it has been shown [17] that the relation between $X_\gamma$ and $X_{-\gamma}$ can also be found if the waveguide possesses one of the following symmetries: reflection symmetry in a plane perpendicular to the waveguide axis; 180 degree rotation symmetry about an axis perpendicular to the waveguide axis; inversion symmetry in a point on the waveguide axis. With the medium tensor in the form of (29) and $\vec{\epsilon}_{tz} = \vec{\epsilon}_{zt}^{\,t}$, $\vec{\mu}_{tz} = \vec{\mu}_{zt}^{\,t}$, the field with a propagation constant $-\gamma$ can be found by the transformation relations of the field under a symmetrical operation. First, considering the waveguide with reflection symmetry, we have

$$\begin{aligned} \mathbf{E}_{-\gamma} &= E_{\gamma,x}(x, y, -z)\hat{x} + E_{\gamma,y}(x, y, -z)\hat{y} \\ &\quad - E_{\gamma,z}(x, y, -z)\hat{z} \\ \mathbf{H}_{-\gamma} &= -H_{\gamma,x}(x, y, -z)\hat{x} - H_{\gamma,y}(x, y, -z)\hat{y} \\ &\quad + H_{\gamma,z}(x, y, -z)\hat{z} \end{aligned} \tag{45}$$

with $E_{\gamma,x}$ representing the $x$ component of the electric field and likewise for the other field components. Second, considering the waveguide with rotation symmetry about the $x$ axis, we have

$$\begin{aligned} \mathbf{E}_{-\gamma} &= E_{\gamma,x}(x, -y, -z)\hat{x} - E_{\gamma,y}(x, -y, -z)\hat{y} \\ &\quad - E_{\gamma,z}(x, -y, -z)\hat{z} \\ \mathbf{H}_{-\gamma} &= H_{\gamma,x}(x, -y, -z)\hat{x} - H_{\gamma,y}(x, -y, -z)\hat{y} \\ &\quad - H_{\gamma,z}(x, -y, -z)\hat{z}. \end{aligned} \tag{46}$$

Next, considering the waveguide with inversion symmetry, we have

$$\begin{aligned} \mathbf{E}_{-\gamma} &= -E_{\gamma,x}(-x, -y, -z)\hat{x} - E_{\gamma,y}(-x, -y, -z)\hat{y} \\ &\quad - E_{\gamma,z}(-x, -y, -z)\hat{z} \\ \mathbf{H}_{-\gamma} &= H_{\gamma,x}(-x, -y, -z)\hat{x} + H_{\gamma,y}(-x, -y, -z)\hat{y} \\ &\quad + H_{\gamma,z}(-x, -y, -z)\hat{z}. \end{aligned} \tag{47}$$

Hence, (41) can be written as

$$\gamma = \frac{\langle \Phi_{-\gamma} | L \Phi_\gamma \rangle}{\langle \Phi_{-\gamma} | \Gamma_z \Phi_\gamma \rangle} \tag{48}$$

with $\Phi_{-\gamma}$ given by (44)–(47). The formulation in (48) has the following properties:

1) It contains second order derivatives in space. Higher order interpolation functions, which are second order differentiable, are commonly used in many problems such as waveguides with convex polygon shapes [18] and surface modes in microstrip [19]. Even with these higher order elements, lack of continuity in the first order derivative between elements may be problematic in the case of second order derivatives in the functional. The second order derivatives can be reduced to first order derivatives using integration by parts, as shown later in this section.
2) The formulation is represented by only transverse fields. The longitudinal components will be given by Faraday's and Ampere's law, represented in (38) and (39), respectively. Hence, the divergence-free condition will be satisfied since **B** or **D** will be represented as the curl of a vector.
3) The permittivity and permeability tensors can have all nine components in the formulation. In reciprocal systems, the material tensors must be symmetric. There are no other restrictions on the elements of the material tensors in (48) as long as the waveguide is bidirectional. The material tensors may be complex, functions of frequency, or functions of position. Hence, (48) can be applied in lossy, inhomogeneous, anisotropic reciprocal problems. Note that if the symmetric conditions (45)–(47) are used, the material tensors will have the same restrictions.
4) The variational expression is in the form of standard generalized eigenvalue equation where the propagation constant appears explicitly as the desired eigenvalue.

Each of the variational formulations proposed previously has more restrictions than (48). These restrictions can be classified into the following three types:

1) variational formulations which are restricted to lossless media [3], [9], [10];
2) variational formulations which only apply to media with the longitudinal part of the medium tensor decoupled from the transverse part [7]–[9];
3) variational formulations which do not enforce the divergence-free condition [3], [6].

Hence, we have derived in (48) for the first time a variational formulation for the propagation constant satisfying the divergence-free condition in lossy inhomogeneous anisotropic yet reciprocal waveguides whose media tensors have full nine components as long as the waveguide is bi-directional. With $X_\lambda^a$ replacing $X_{-\lambda}$ in (41), the formulation in (41) is variational even in lossy nonreciprocal problems as long as the waveguide and its adjoint waveguide is mutually bi-directional, which is equivalent to the condition that the matrix $D$ in (44) is nonsingular.

For some applications, it is not preferable or necessary to avoid the previously mentioned restrictions. We will derive the corresponding variational formulations for such applications. Even with the same restrictions with a decoupled longitudinal part of the medium tensor, the following derived formulations, which can be used in more general problems or yield a better approximation for the propagation constant, will have some advantages over the corresponding formulations in the literature.

When the transverse part of the media tensors are decoupled from the longitudinal part, i.e.,

$$\overline{\overline{\epsilon}} = \begin{bmatrix} \overline{\epsilon}_{tt} & 0 \\ 0 & \epsilon_{zz} \end{bmatrix} \qquad \overline{\overline{\mu}} = \begin{bmatrix} \overline{\mu}_{tt} & 0 \\ 0 & \mu_{zz} \end{bmatrix} \tag{49}$$

we can derive a variational form for $\gamma^2$ in terms of only $\mathbf{h}$. By using the relationship $\nabla \cdot \mathbf{B} = 0$, we can represent $H_z$ in terms of $\mathbf{h}$

$$H_z = \frac{\hat{z} \cdot \overline{\overline{\mu}}^{-1} \cdot \hat{z}\nabla_t \cdot \overline{\overline{\mu}} \cdot \mathbf{h}}{j\gamma}. \tag{50}$$

By using (3) with (50), to represent $H_z$ in terms of $\mathbf{h}$, and premultiplying (3) with $\overline{\overline{\epsilon}} \cdot \hat{z}\times$, we get [8]

$$\gamma^2 \hat{z} \times \mathbf{h} - \hat{z} \times \nabla_t \hat{z} \cdot \overline{\overline{\mu}}^{-1} \cdot \hat{z}\nabla_t \cdot \overline{\overline{\mu}} \cdot \mathbf{h} - \omega^2 \overline{\overline{\epsilon}} \cdot \hat{z} \times \overline{\overline{\mu}} \cdot \mathbf{h} - \overline{\overline{\epsilon}} \cdot \hat{z} \times \nabla_t \times \overline{\overline{\epsilon}}^{-1} \cdot \nabla_t \times \mathbf{h} = 0. \tag{51}$$

Note that (51) is a form of (6), which can be expressed as $A \cdot \mathbf{h}_\gamma - \gamma^2 B \cdot \mathbf{h}_\gamma = 0$. Hence, from (15) we have

$$\gamma^2 = \frac{\langle \mathbf{h}_\gamma^{\mathbf{a}} | A \cdot \mathbf{h}_\gamma \rangle}{\langle \mathbf{h}_\gamma^{\mathbf{a}} | B \cdot \mathbf{h}_\gamma \rangle}. \tag{52}$$

Using the result in [8], we have $\mathbf{h}_\gamma^{\mathbf{a}} = \mathbf{e}_\gamma$. The transverse electric field, $\mathbf{e}_\gamma$, must satisfy

$$\mathbf{e}_\gamma = \frac{-\omega \cdot \hat{z} \times \overline{\overline{\mu}} \cdot \mathbf{h}}{\gamma} - \frac{\nabla_t \frac{1}{\epsilon_{zz}} \nabla_t \cdot (\hat{z} \times \mathbf{h})}{\gamma\omega}. \tag{53}$$

Thus, after some manipulation of (53), we have for

$$\gamma^2 = \frac{\langle \mathbf{e}_\gamma | A \cdot \mathbf{h}_\gamma \rangle}{\langle \mathbf{e}_\gamma | B \cdot \mathbf{h}_\gamma \rangle} = \frac{\langle -\omega^2 \cdot \hat{z} \times \overline{\overline{\mu}} \cdot \mathbf{h} - \nabla_t \frac{1}{\epsilon_{zz}} \nabla_t \cdot (\hat{z} \times \mathbf{h}) | A \cdot \mathbf{h}_\gamma \rangle}{\langle -\omega^2 \cdot \hat{z} \times \overline{\overline{\mu}} \cdot \mathbf{h} - \nabla_t \frac{1}{\epsilon_{zz}} \nabla_t \cdot (\hat{z} \times \mathbf{h}) | \hat{z} \times \mathbf{h}_\gamma \rangle}. \tag{54}$$

Equation (54) is in the form of standard generalized eigenvalue equation where the propagation constant appears explicitly as the desired eigenvalue. The longitudinal components of electric and magnetic field will be given by the divergence-free condition. The divergence-free condition is enforced in (54). Davies has given a formulation similar to (54) which does not apply to media with inhomogeneous permeability [7]. On the other hand, (54) works for the case of media with inhomogeneous permeability and permittivity as $A$ contains spatial derivatives of permeability.

The formulation given in (54) contains second order derivatives in space. In (54) the second order derivatives can be reduced to first order by integration by parts if (54) is represented in terms of both $\mathbf{e}_\gamma$ and $\mathbf{h}_\gamma$. In reciprocal systems, the material tensors must be symmetric. There are no other restrictions on the elements of the material tensors. They may be complex, functions of frequency or functions of position.

We can also give the variational expression in terms of full magnetic field. It will be also possible to reduce the second order derivative in space to a first order derivative. In the following derivation, we again assume that the transverse part of the media tensors are decoupled from the longitudinal part, as in (49). Using (3), (5), (50) and the adjoint field solution [5]

$$\mathbf{H}_\gamma^{\mathbf{a}} = \mathbf{h}_\gamma - H_z \hat{z} \tag{55}$$

we have

$$\begin{aligned} \gamma^2 = &(-2\langle \hat{z} \times \mathbf{h}_\gamma | \overline{\overline{\epsilon}}^{-1} \cdot \nabla_t \times \hat{z} \frac{1}{\mu_{zz}} \nabla_t \cdot \overline{\overline{\mu}} \cdot \mathbf{h}_\gamma \rangle \\ &- \langle \nabla_t \times \mathbf{h}_\gamma | \overline{\overline{\epsilon}}^{-1} \cdot \nabla_t \times H_z \hat{z} \rangle \\ &+ \langle \nabla_t \times H_z \hat{z} | \overline{\overline{\epsilon}}^{-1} \cdot \nabla_t \times H_z \hat{z} \rangle + \langle \mathbf{h}_\gamma | \omega^2 \overline{\overline{\mu}} \cdot \mathbf{h}_\gamma \rangle \\ &- \langle H_z \hat{z} | \omega^2 \overline{\overline{\mu}} \cdot H_z \hat{z} \rangle) / \langle \hat{z} \times \mathbf{h}_\gamma | \overline{\overline{\epsilon}}^{-1} \cdot \hat{z} \times \mathbf{h}_\gamma \rangle. \end{aligned} \tag{56}$$

Davies [6] has derived a variational formulation for the propagation constant in terms of full magnetic field vector. However, (56) will yield a better approximation for the propagation constant since the Euler equation of Davies' formulation (37) in [6] is not the vector magnetic field equation [12]. (56) is in the form of standard generalized eigenvalue equation where the propagation constant appears explicitly as the desired eigenvalue. Spurious modes can occur in using the functional given in (56), since the vector that satisfies (56) does not automatically satisfy the divergence-free condition. The penalty parameter method [11] can be used to remove the spurious modes. The second order derivative in (56) can be reduced to first order using integration by parts. The elements of the material tensors may be complex, functions of frequency or functions of position.

If the system is lossy and nonreciprocal, the derivation of a adjoint solution must be investigated for each specific problem. We give an example for such waveguide.

With the choice of a real inner product, the adjoint field must satisfy Maxwell's equations with transposed permittivity and permeability tensor and with propagation constant $-\gamma$ [15]. This requires that the original and adjoint waveguide be mutually bi-directional [17]. Assuming that the media tensors are of the form

$$\bar{\bar{\epsilon}} = \begin{bmatrix} \bar{\bar{\epsilon}}_{tt} & \vec{\epsilon}_{tz} \\ -\vec{\epsilon}^{\,t}_{tz} & \epsilon_{zz} \end{bmatrix}$$
$$\bar{\bar{\mu}} = \begin{bmatrix} \bar{\bar{\mu}}_{tt} & \vec{\mu}_{tz} \\ -\vec{\mu}^{\,t}_{tz} & \mu_{zz} \end{bmatrix} \tag{57}$$

with $\bar{\bar{\epsilon}}_{tt}$ and $\bar{\bar{\mu}}_{tt}$ symmetric, the adjoint field will be mutually bi-directional to the original waveguide and can be written as [5]

$$\begin{aligned} \mathbf{E}^{\mathbf{a}}_{\gamma} &= -\mathbf{e}_{\gamma} + E_z\hat{z} \\ \mathbf{H}^{\mathbf{a}}_{\gamma} &= -\mathbf{h}_{\gamma} + H_z\hat{z}. \end{aligned} \tag{58}$$

Hence

$$\gamma = \frac{\langle \Phi^a_\gamma | L\Phi_\gamma \rangle}{\langle \Phi^a_\gamma | \Gamma_z \Phi_\gamma \rangle} \tag{59}$$

with

$$\Phi^a_\gamma = \begin{bmatrix} -\mathbf{e}_\gamma \\ j\mathbf{h}_\gamma \end{bmatrix}. \tag{60}$$

The material tensor of transversely DC magnetized, low loss, magnetically-saturated ferrites is of the form (57) [16]. The formulation given in (59) works for a lossy nonreciprocal media as long as the adjoint solution exists. For this specific waveguide problem, Chen has derived a variational formulation [(43) in [5]], which does not satisfy the divergence-free condition, to demonstrate the application of his variational technique. On the other hand, the divergence-free condition will be satisfied in using (59) since the longitudinal component $E_z$ and $H_z$ are given by divergence condition using transverse field and from solution of (59).

## V. Reduction of the Spatial Derivative of the Variational Formulation

In the last part of this paper, the second order derivative term in the variational formulation will be reduced to first order derivative. By the utilization of the vector identity

$$\int A \cdot \nabla_t \times B\, ds = \oint B \times A \cdot \hat{n}\, dl + \int B \cdot \nabla_t \times A\, ds \tag{61}$$

where $\hat{n}$ is the unit normal vector, the second order derivative term in the functional can be reduced to first order derivative. In addition, the boundary line integral in (61) vanishes with the application of suitable boundary conditions. For example, the functional in (48) has second order derivatives in the $L_{11}$ and $L_{22}$ terms. With the adjoint solution $\Phi_{-\gamma}$, the second order derivative term in $L_{11}$ can be represented as

$$\int \mathbf{e}_{-\gamma} \cdot \frac{1}{\omega} \nabla_t \times \hat{z} \frac{1}{\mu_{zz}} \hat{z} \times \nabla_t \cdot \mathbf{e}_\gamma\, ds. \tag{62}$$

Using (61), (62) can be transformed to

$$\int \nabla_t \times \mathbf{e}_{-\gamma} \cdot \frac{1}{\omega} \hat{z} \frac{1}{\mu_{zz}} \hat{z} \times \nabla_t \cdot \mathbf{e}_\gamma\, ds + \oint \hat{n} \cdot \mathbf{e}_{-\gamma} \times \frac{1}{\omega} \hat{z} \frac{1}{\mu_{zz}} \hat{z} \times \nabla_t \cdot \mathbf{e}_\gamma\, dl. \tag{63}$$

Similarly, the second derivative term from $L_{22}$ can be reduced to first order derivative. The term in (A3) involving line integrals vanish if there is a pec or pmc at the boundary or if the boundary is at infinity [8]. For the variational expression in (54) with the substitution of (51), the second order derivative terms can be represented as

$$\int \mathbf{e} \cdot \hat{z} \times \nabla_t \hat{z} \cdot \bar{\bar{\mu}}^{-1} \cdot \hat{z} \nabla_t \cdot \bar{\bar{\mu}} \cdot \mathbf{h}\, ds \tag{64}$$

and

$$\int \mathbf{e} \cdot \bar{\bar{\epsilon}} \cdot \hat{z} \times \nabla_t \times \bar{\bar{\epsilon}}^{-1} \nabla_t \times \mathbf{h}\, ds. \tag{65}$$

Using (61), (64) can be transformed to

$$-\int \nabla_t \times \mathbf{e} \cdot \hat{z}\hat{z} \cdot \bar{\bar{\mu}}^{-1} \cdot \hat{z} \nabla_t \cdot \bar{\bar{\mu}} \cdot \mathbf{h}\, ds - \oint \hat{n} \cdot \mathbf{e} \times \hat{z}\hat{z} \cdot \bar{\bar{\mu}}^{-1} \cdot \hat{z} \nabla_t \cdot \bar{\bar{\mu}} \cdot \mathbf{h}\, dl. \tag{66}$$

Using (A1), (A7) can be transformed to

$$-\int \bar{\bar{\epsilon}}^{-1} \nabla_t \times \mathbf{h} \cdot \nabla_t \times \hat{z} \times \bar{\bar{\epsilon}} \cdot \mathbf{e}\, ds - \oint \hat{n} \cdot \bar{\bar{\epsilon}}^{-1} \nabla_t \times \mathbf{h} \times \hat{z} \times \bar{\bar{\epsilon}} \cdot \mathbf{e}\, dl. \tag{67}$$

Again, the terms in (66) and (67) involving line integrals vanish with a pec or pmc boundary or if the boundary is at infinity. For the variational expression in (56), the second order derivative term can be represented as

$$-2 \int \hat{z} \times \mathbf{h}_\gamma \cdot \bar{\bar{\epsilon}}^{-1} \cdot \nabla_t \times \hat{z} \frac{1}{\mu_{zz}} \nabla_t \cdot \bar{\bar{\mu}} \cdot \mathbf{h}_\gamma\, ds. \tag{68}$$

Using (61), (68) can be transformed to

$$-2 \int \hat{z} \frac{1}{\mu_{zz}} \nabla_t \cdot \bar{\bar{\mu}} \cdot \mathbf{h}_\gamma \cdot \nabla_t \times \bar{\bar{\epsilon}}^{-1} \cdot \hat{z} \times \mathbf{h}_\gamma\, ds - 2 \oint \hat{z} \frac{1}{\mu_{zz}} \nabla_t \cdot \bar{\bar{\mu}} \cdot \mathbf{h}_\gamma \times \bar{\bar{\epsilon}}^{-1} \cdot \hat{z} \times \mathbf{h}_\gamma \cdot \hat{n}\, dl. \tag{69}$$

## VI. Conclusion

We have derived a variational formulation for the propagation constant satisfying the divergence-free condition in lossy inhomogeneous anisotropic reciprocal or nonreciprocal waveguides whose media tensors have all nine components. Several variational formulations for the propagation constant for waveguides where the longitudinal part of the medium tensor is decoupled from the transverse part have already been proposed in the literature. However, applied in finite element methods, the new formulations will have several advantages over previously published formulations. All the variational expressions are in the form of standard generalized eigenvalue equations, where the propagation constant appears explicitly as the desired eigenvalue. It is also shown that for a general lossy nonreciprocal problem the variational functional exists only if the original and adjoint waveguide are mutually bi-directional. On the other hand, for a general lossy reciprocal problem the variational functional exists only if the waveguide is bi-directional.

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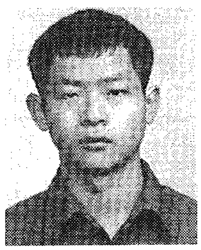


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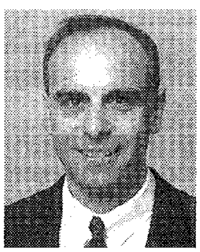


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